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EEN320 - Power Systems I (Συστήματα Ισχύος Ι)

Part 2: Single-phase and three-phase AC systems

https://sps.cut.ac.cy/courses/een320/

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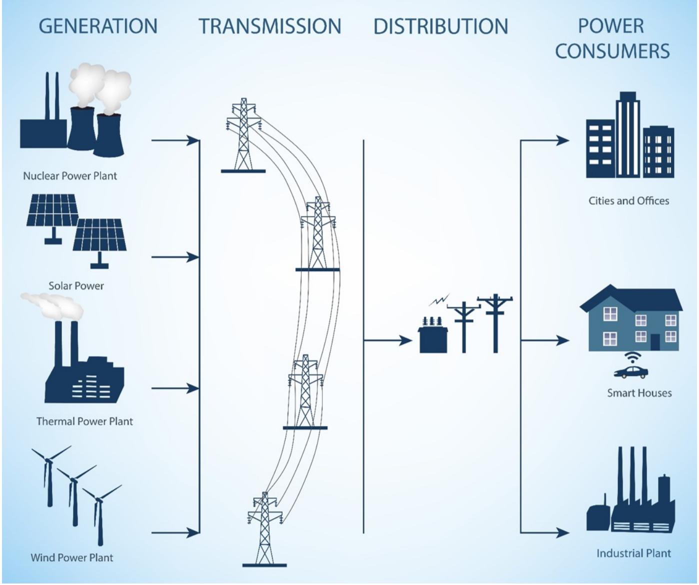
Recap from last lecture - Overview power system



Learning objectives

After this part of the lecture and additional reading, you should be able to . . .

1. . . . calculate complex voltages, currents and powers in single-phase systems using phasors;
2. . . . explain standard configurations of three-phase power systems;
3. . . . calculate complex voltages, currents and powers in balanced three-phase systems using the concepts of phasors.

Review: Calculations with complex numbers

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- Consider a complex number $\underline{z} = a + jb$, where $a \in \mathbb{R}$, $b \in \mathbb{R}$ and j is the imaginary unit satisfying $j^2 = -1$
- Complex conjugate of $\underline{z}$ is defined as $\underline{z}^* = a - jb$
- Absolute value $|\underline{z}|$ and argument ϕ of $\underline{z}$ are defined as

$$|\underline{z}| = \sqrt{a^2 + b^2}, \quad \varphi = \arctan\left(\frac{b}{a}\right)$$

  (if $a > 0$ which is usually the case in power systems; otherwise use atan2-function to determine φ)
- Polar representation of $\underline{z}$ via Euler's formula

$$\underline{z} = |\underline{z}|e^{j\varphi} = |\underline{z}|\underline{/\varphi}$$

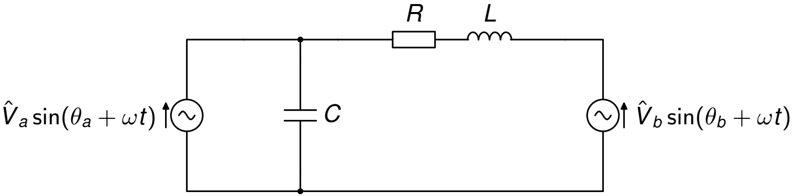
- Conversion of radians to degrees: 1 rad = 57.3 degrees

Outline

1. **Power in single-phase AC systems**
2. **Electrical quantities as phasors in the complex plane**
3. **Complex apparent, real and reactive power in single-phase systems**
4. **Conservation of complex apparent power**
5. **Balanced three-phase AC systems**
   - Balanced three-phase AC waveforms, circuits and systems
   - Y- and Delta-configurations of three-phase AC systems
6. **Power in balanced three-phase AC systems**
7. **Advantages of three-phase over 3 single-phase systems**

Some general assumptions

- In this part of the lecture, we will consider single- and three-phase AC networks
- We will do this under the following assumptions
  - The network only contains passive elements (R, L, C) and purely sinusoidal current and voltage sources of identical frequencies
  - The network is in steady-state
- Example single-phase circuit:



1 Outline

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1 Single-phase AC waveforms

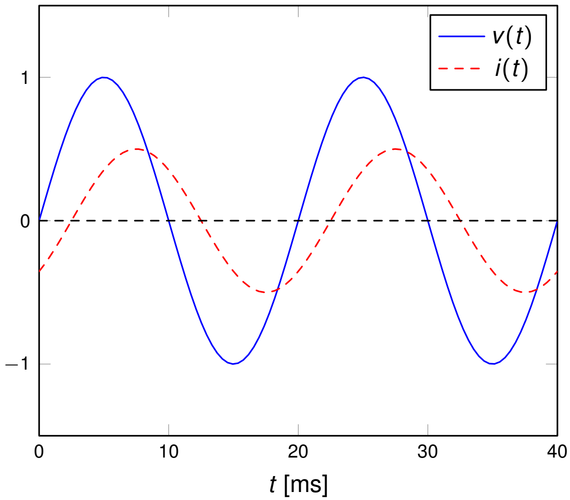
- Instantanteous values (στιγμιαίες τιμές) of voltage and current at a network element given by

$$v(t) = \hat{V}\cos(\omega t)$$
$$i(t) = \hat{I}\cos(\omega t - \varphi)$$

- $\hat{V}$ and $\hat{I}$ are the *amplitudes* (μέγιστη τιμή) of the respective waveforms (κυμματομορφές)
- φ is the *phase shift* (διαφορά φάσης) between voltage and current
- $\omega = 2\pi f$, where f is the stationary (στάσιμη) electrical frequency of the network (e.g., 50 Hz in Europe; 60 Hz in US)
- Instead of peak-to-peak amplitudes $\hat{V}$ and $\hat{I}$, often root-mean-square (also called effective) amplitudes V and I used

$$\hat{V} = \sqrt{2}V, \quad \hat{I} = \sqrt{2}I$$

1 Single-phase AC waveforms - Example (time on x-axis)

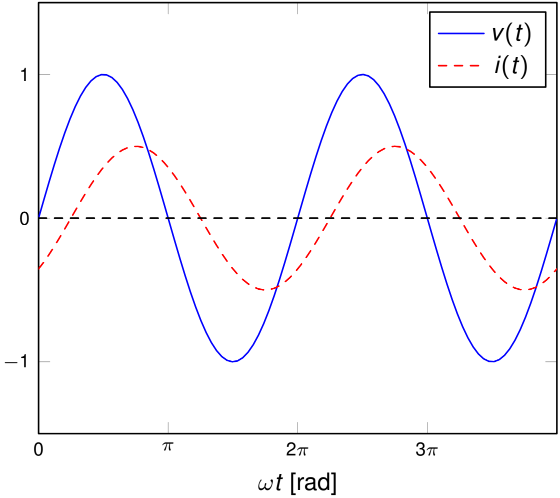


$$\omega = 2\pi f$$

$$f = 50 \text{ Hz}$$

$$\varphi = \frac{\pi}{4}$$

1 Single-phase AC waveforms - Example (phase on x-axis)



$$\omega = 2\pi f$$

$$f = 50 \text{ Hz}$$

$$\varphi = \frac{\pi}{4}$$

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- Instantaneous single-phase AC power

$$p(t) = v(t)i(t) = \hat{V}\hat{I}\cos(\omega t)\cos(\omega t - \varphi)$$

- By using the trigonometric identities

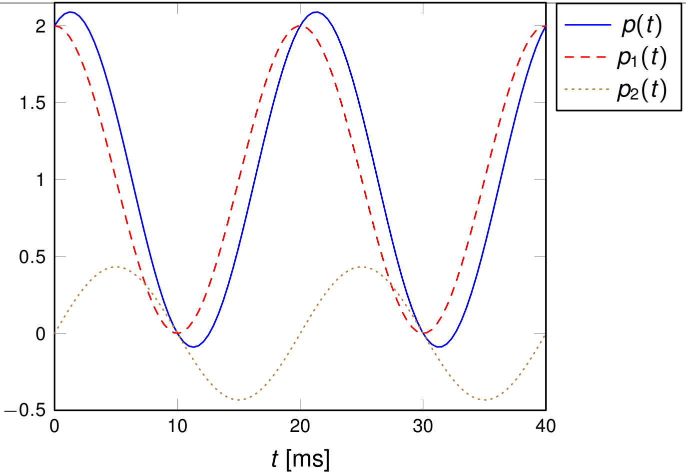
$$\cos(\omega t - \varphi) = \cos(\omega t)\cos(\varphi) + \sin(\omega t)\sin(\varphi)$$

$$\cos^2(\omega t) = \frac{1}{2}(1 + \cos(2\omega t)),\ \sin(\omega t)\cos(\omega t) = \frac{1}{2}\sin(2\omega t)$$

we have that

$$\begin{aligned} p(t) &= \hat{V}\hat{I}\cos(\omega t)\left(\cos(\omega t)\cos(\varphi) + \sin(\omega t)\sin(\varphi)\right) \\ &= \hat{V}\hat{I}\left((\cos(\omega t))^2\cos(\varphi) + \cos(\omega t)\sin(\omega t)\sin(\varphi)\right) \\ &= \hat{V}\hat{I}\left(\frac{1}{2}(1 + \cos(2\omega t))\cos(\varphi) + \frac{1}{2}\sin(2\omega t)\sin(\varphi)\right) \\ &= \frac{1}{2}\hat{V}\hat{I}\cos(\varphi)(1 + \cos(2\omega t)) + \frac{1}{2}\hat{V}\hat{I}\sin(\varphi)\sin(2\omega t) \end{aligned}$$

1 Individual components of instantaneous power



$$p(t) = \underbrace{\frac{1}{2}\hat{V}\hat{I}\cos(\varphi)(1+\cos(2\omega t))}_{p_1(t)} + \underbrace{\frac{1}{2}\hat{V}\hat{I}\sin(\varphi)\sin(2\omega t)}_{p_2(t)}$$

1 Active power in single-phase systems

- $p_1(t)$ is oscillating between 0 and $\hat{V}\hat{I}\cos(\varphi)$
- $p_1(t)$ never changes sign $\rightarrow$ always flows in same direction
- The average value over time of $p_1(t)$ is

$$P = \frac{1}{T}\int_0^T p(t)dt = \frac{1}{2}\hat{V}\hat{I}\cos(\varphi) = VI\cos(\varphi)$$

- P is called *active power* (ενεργός ισχύς)
- P is the only "useful" component of $p(t)$
- $\cos(\varphi)$ is called *power factor* (συντελεστής ισχύος)
- φ is called *power factor angle*
- The unit of P is the Watt [W]

- $p_2(t)$ is oscillating between $\pm \hat{V}\hat{I}\sin(\varphi)$
- Average value over time of $p_2(t)$ is zero $\rightarrow$ no ”useful” work
- The amplitude of the waveform $p_2(t)$ is

$$Q = \frac{1}{2}\hat{V}\hat{I}\sin(\varphi) = VI\sin(\varphi)$$

- Q is called *reactive power* (άεργος ισχύς)
- The unit of Q is the Volt-Ampere-reactive [Var] (also used: [var])

1 Reactive power in *RLC* circuits

- In *RLC* circuits, Q appears because of the presence of inductors and capacitors
- In fact, Q is the time derivative of the energy stored in inductors and capacitors
- These elements continuously accumulate and release energy
- They never release more energy than they have accumulated (that's why they are also called *passive* elements)

→ The energy is always nonnegative

- Important: in general networks, it is far more difficult to associate a clear physical meaning to Q

2 Outline

2 Phasors in electrical power systems

- Sinusoidal waveforms can also be represented by *phasors* in the complex plane
- Phasors are very popular in electric power systems
- Main reasons: simplify visualisation and calculation of electrical networks
- This is very useful for analysis, design and operation of power systems

2 Definition of a phasor (φασιθέτης)

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- Consider

$$x(t) = \hat{X}\cos(\omega t + \theta)$$

- Via Euler's Formula, we define the phasor corresponding to $x(t)$ as[1]

$$\underline{X} = \frac{\hat{X}}{\sqrt{2}}\left(\cos(\theta) + j\sin(\theta)\right) = \underbrace{X\left(\cos(\theta) + j\sin(\theta)\right)}_{\text{trigonometric form}} = \underbrace{Xe^{j\theta}}_{\text{exponential form}}$$

- Then

$$x(t) = \sqrt{2}\Re\{\underline{X}e^{j\omega t}\},$$

i.e., momentary value of $x(t)$ corresponds to real part of the phasor $\underline{X}$ rotating at angular speed ω

- Alternative common notation for a phasor

$$\underline{X} = Xe^{j\theta} = \underbrace{X\angle\theta}_{\text{angular form}}$$

[1] Here j denotes the imaginary unit.

2 Voltages and currents as complex phasors

- Phasors of voltage and current

$$\underline{V} = V\left(\cos(\varphi_v) + j\sin(\varphi_v)\right) = Ve^{j\varphi_v} = V\angle\varphi_v$$
$$\underline{I} = I\left(\cos(\varphi_i) + j\sin(\varphi_i)\right) = Ie^{j\varphi_i} = I\angle\varphi_i$$
$$\varphi = \varphi_v - \varphi_i$$

- Note: as we have assumed stationary conditions, it suffices to use $\underline{X}$ to describe $x(t)$ for network calculations

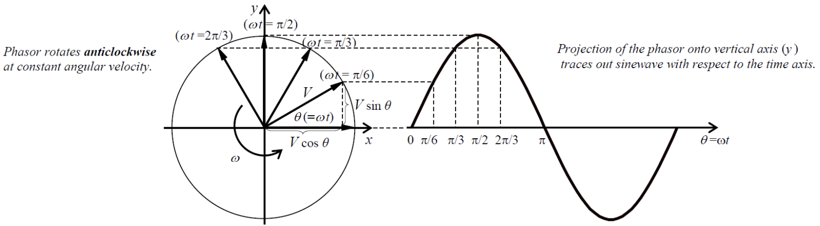
  Why? Because the term $e^{j\omega t}$ cancels out, whenever multiplying two complex quantities

$$\left(\underline{V}e^{j\omega t}\right)\left(\underline{I}e^{j\omega t}\right)^* = \underline{V}\,\underline{I}^* e^{j\omega t}e^{-j\omega t} = \underline{V}\,\underline{I}^*,$$

  where the operator * denotes complex conjugation

2 Visualization of a phasor

$$\underline{V} = Ve^{j\theta} = \underbrace{V\underline{/\theta}}_{\text{angular form}}$$



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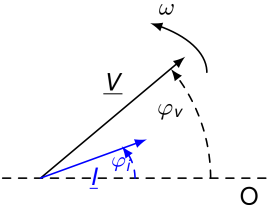
- starting from the origin $0 + j0$
- projection on the real axis is $\frac{1}{\sqrt{2}}v(t)$
- the phasor is the position at $t = 0$ of the rotating vector

2 Phasor diagrams

- Phasor diagrams (φασικά διαγράμματα): A graphical representation of the phasors



3 Outline

3 Complex apparent power in single-phase systems

- Now, we can introduce a third important quantity in power systems - the complex apparent power

$$\underline{S} = \underline{V}\,\underline{I}^{*} = VIe^{j(\varphi_v - \varphi_i)} = VIe^{j\varphi} = VI(\cos(\varphi) + j\sin(\varphi))$$

- Remember that $\varphi = \varphi_v - \varphi_i$ is called the power factor angle and it's connected to power factor as $PF = cos\varphi$
- The absolute value of the complex apparent power is called apparent power S

$$S = |\underline{S}| = VI$$

- The unit of $\underline{S}$ and S is Volt-Ampere [VA]
- Apparent power used to dimension equipment

$$S = VI \quad \Rightarrow \quad S = P \quad \text{if} \quad \varphi = 0$$

3 Relation between $\underline{S}$, P and Q

- Active power P corresponds to real part of $\underline{S}$

$$P = \Re\{\underline{S}\} = VI\cos(\varphi)$$

- Reactive power Q corresponds to imaginary part of $\underline{S}$

$$Q = \Im\{\underline{S}\} = VI\sin(\varphi)$$

- Hence

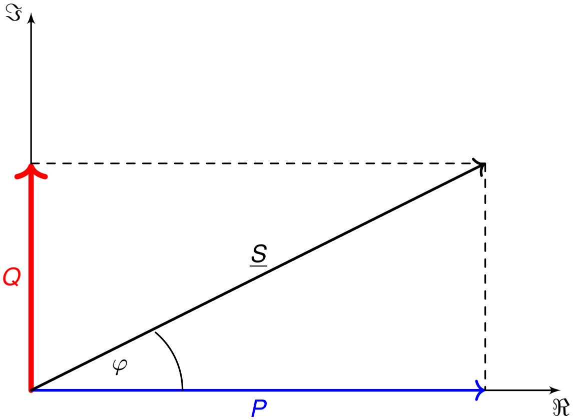
$$\underline{S} = P + jQ$$

and

$$S = |\underline{S}| = \sqrt{P^2 + Q^2}$$

$p(t)$, P	Watt	W	kW, MW
Q	Var	(VAr, Var, var)	kvar, Mvar
S	Volt-Ampere	VA	kVA, MVA

3 Power triangle in the complex plane

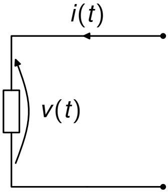


Power factor

$$\cos(\varphi) = \frac{P}{|\underline{S}|} = \frac{P}{\sqrt{P^2 + Q^2}}$$

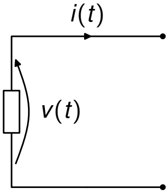
- For network calculations, it is important to determine whether an element absorbs or delivers power
- There are 2 conventions:

Load convention (standard)



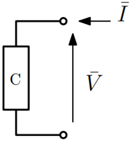
Current counted positively if enters circuit element ”by head” of voltage arrow

Generator convention



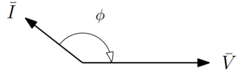
Current counted positively if leaves circuit element ”by head” of voltage arrow

3 Load convention

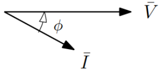


$P = VI\cos\phi$ is the active power consumed by C

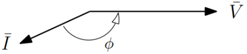
$Q = VI\sin\phi$ is the reactive power consumed by C



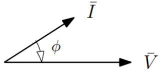
C produces active power
produces reactive power



C consumes active power
consumes reactive power

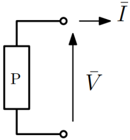


C produces active power
consumes reactive power



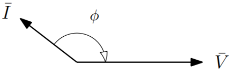
C consumes active power
produces reactive power

3 Generator convention

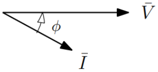


$P = VI\cos\phi$ is the active power produced by P

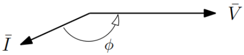
$Q = VI\sin\phi$ is the reactive power produced by P



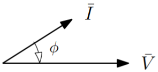
P consumes active power
consumes reactive power



P produces active power
produces reactive power



P consumes active power
produces reactive power

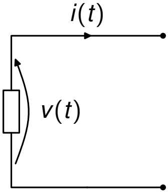


P produces active power
consumes reactive power

3 Generator convention

The power factor can be also defined for generators:

$$\cos\varphi = \frac{P_g}{\sqrt{P_g^2 + Q_g^2}}$$



- a generator can produce or absorb reactive power
- whether it absorbs or consumes is not shown by the power factor
- hence, sometimes the value of the power factor is followed by
  - ”inductive” if reactive power is produced (the generator feeds an inductive load)
  - ”capacitive” if reactive power is consumed (the generator feeds a capacitive load)
- less ambiguous:

$$\tan\varphi = \frac{Q_g}{P_g}$$

3 Example: Active and reactive power calculation

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$$\underline{V} = 1\angle 0 \qquad \underline{I} = 0.5\angle -\frac{\pi}{6} \qquad \omega = 2\pi 50 = 314 \text{ rad/s}$$

Task.

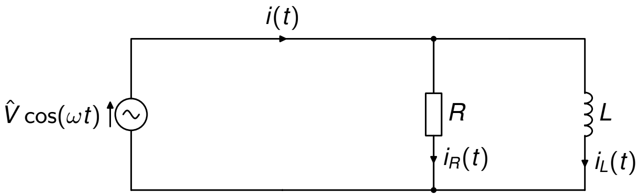
- Given the above phasors, write the time-domain sinusoidal equations.
- Compute the power factor angle, power factor, active, reactive, and apparent powers.

3 Example: Active and reactive power calculation

Solution.

$$v(t) = \sqrt{2}V\cos(\omega t) = \sqrt{2}\cos(\omega t) \quad i(t) = \sqrt{2}0.5\cos\left(\omega t - \frac{\pi}{6}\right)$$

$$\begin{aligned}
\varphi &= \frac{\pi}{6} \\
\cos(\varphi) &= 0.866 \\
P &= VI\cos(\varphi) = 0.5 \cdot 0.866 = 0.433 \\
Q &= VI\sin(\varphi) = 0.5 \cdot 0.5 = 0.25 \\
S &= VI = 0.5 = \sqrt{P^2 + Q^2}
\end{aligned}$$



Task.

- Given the above electrical network and waveform characteristics, calculate the stationary power consumption of the resistor together with the inductor.
- At first, use the time domain expressions for v and i. Then use the phasors $\underline{V}$ and $\underline{I}$.
- Determine the power factor of the circuit.

Solution.

1) Using time domain expressions

From KCL

$$i(t) = i_R(t) + i_L(t) = \frac{\hat{V}}{R}\cos(\omega t) + \frac{\hat{V}}{\omega L}\cos\left(\omega t - \frac{\pi}{2}\right)$$

Hence, instantaneous power

$$\begin{aligned} p(t) = v(t) \cdot i(t) &= \frac{\hat{V}^2}{R}\cos^2(\omega t) + \frac{\hat{V}^2}{\omega L}\cos\left(\omega t - \frac{\pi}{2}\right)\cos(\omega t) \\ &= \frac{1}{2}\frac{\hat{V}^2}{R}(1 + \cos(2\omega t)) + \frac{1}{2}\frac{\hat{V}^2}{\omega L}\sin(2\omega t) \\ &= \underbrace{\frac{V^2}{R}(1 + \cos(2\omega t))}_{p_R(t)} + \underbrace{\frac{V^2}{\omega L}\sin(2\omega t)}_{p_L(t)} \end{aligned}$$

Note: instantaneous power consists of two components - power on resistance $p_R(t)$ and power on inductance $p_L(t)$

Solution.

From previous considerations, we have that active and reactive power are given by

$$P = \frac{V^2}{R}, \qquad Q = \frac{V^2}{\omega L} = \frac{V^2}{X_L}$$

2) Using phasors

Complex current

$$\underline{I} = \frac{\underline{V}}{\underline{Z}} = \underline{V}\left(\frac{1}{R} + \frac{1}{j\omega L}\right)$$

Complex apparent power (setting $\underline{V} = V\underline{/0^\circ}$)

$$\underline{S} = \underline{V}\,\underline{I}^* = \underline{V}\,\underline{V}^*\left(\frac{1}{R} + j\frac{1}{\omega L}\right) = \frac{V^2}{R} + j\frac{V^2}{\omega L}$$

Active and reactive power

$$P = \Re(\underline{S}) = \frac{V^2}{R}, \qquad Q = \Im(\underline{S}) = \frac{V^2}{\omega L} = \frac{V^2}{X_L}$$

Solution.

Note:

$$Q = \Im(\underline{S}) = \frac{V^2}{\omega L} = \frac{V^2}{X_L} > 0$$

$\rightarrow$ Inductance "consumes" reactive power

Power factor

$$\cos(\varphi) = \frac{P}{S}$$

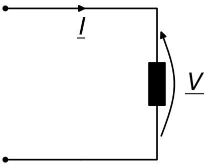
We have that

$$S = V^2 \sqrt{\frac{1}{R^2} + \frac{1}{X_L^2}}$$

Thus

$$\cos(\varphi) = \frac{P}{S} = \frac{V^2}{R} \frac{1}{V^2 \sqrt{\frac{1}{R^2} + \frac{1}{X_L^2}}} = \frac{1}{\sqrt{1 + \frac{R^2}{X_L^2}}}$$

Question: Does the PF depend on the voltage in this case?



	Resistance R	Inductance L	Capacitance C
φ	0	$\frac{\pi}{2}$	$-\frac{\pi}{2}$
P	$RI^2 = \frac{V^2}{R}$	0	0
Q	0	$\omega L I^2 = \frac{V^2}{\omega L}$	$-\frac{I^2}{\omega C} = -\omega C V^2$

$\rightarrow$ Inductor ”consumes” reactive power; power factor of inductive load is said to be *lagging*, because current lags voltage ($\varphi > 0$)

$\rightarrow$ Capacitor ”produces” reactive power; power factor of capacitive load is said to be *leading*, because current leads voltage ($\varphi < 0$)

3 Complex impedance and admittance

- Inductive reactance (επαγωγική αντίδραση) $X = \omega L$ [Ω]
- Capacitive reactance (χωρητική αντίδραση) $X = -\frac{1}{\omega C}$ [Ω]
- Complex impedance (σύνθετη αντίσταση)

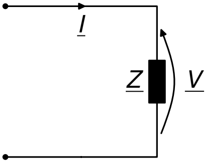
$$\underline{Z} = R + jX \ [\Omega]$$

- Complex admittance (σύνθετη αγωγιμότητα)

$$\begin{aligned}\underline{Y} = \frac{1}{\underline{Z}} = \frac{1}{R+jX} &= \frac{1}{R+jX} \cdot \frac{R-jX}{R-jX} \\ &= \frac{R}{R^2+X^2} + j\frac{-X}{R^2+X^2} \\ &= G + jB \ [\mathrm{S}]\end{aligned}$$

- $G = \frac{R}{R^2+X^2}$ [S] is called conductance (αγωγιμότητα)
- $B = \frac{-X}{R^2+X^2}$ [S] is called susceptance (επαγωγική ή χωρητική επιδεκτικότητα)
- S=siemens; SI unit of conductance, susceptance and admittance

3 Power expressions for one-ports described by a complex impedance



Via impedance

$$\underline{V} = \underline{Z}\,\underline{I}$$

$$\underline{Z} = R + jX$$

$$\underline{S} = \underline{V}\,\underline{I}^* = \underline{Z}\,\underline{I}\,\underline{I}^* = \underline{Z} I^2$$

$$P = \Re\{\underline{S}\} = RI^2$$

$$Q = \Im\{\underline{S}\} = XI^2$$

Via admittance

$$\underline{I} = \underline{Y}\,\underline{V}$$

$$\underline{Y} = G + jB$$

$$\underline{S} = \underline{V}\,\underline{I}^* = \underline{V}(\underline{Y}\,\underline{V})^* = \underline{Y}^* V^2$$

$$P = \Re\{\underline{S}\} = GV^2$$

$$Q = \Im\{\underline{S}\} = -BV^2$$

3 Practical implications of the power factor $\cos(\varphi)$

- $\cos(\varphi) = \frac{P}{|S|} = \frac{P}{\sqrt{P^2+Q^2}} = \frac{P}{VI}$

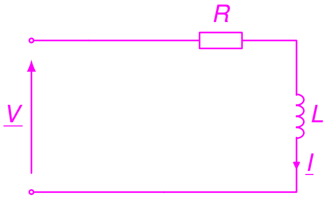
$\Leftrightarrow$ $I = \frac{P}{V\cos(\varphi)}$

$\rightarrow$ For the same useful power P and a fixed voltage V, the smaller the power factor $\cos(\varphi)$ the larger the current I

- Or, equivalently, the larger the phase angle between the voltage and current waveforms, the smaller the power factor
- This has important practical implications!
- For the same useful power P and a fixed voltage V, the larger the reactive power consumed or produced by the load
  - the larger the current I
  - need lines of higher current capacity $\rightarrow$ more costly investment!
  - get higher losses RI^2 in the lines $\rightarrow$ more costly operation!

3 Reactive power compensation (1)

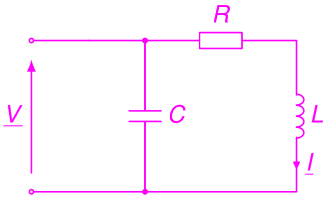
- Most loads *consume* reactive power
- Frequent solution: try to bring power factor closer to 1 by producing reactive power close to (or directly at) the load



$$\cos(\varphi) = \frac{P}{\sqrt{P^2 + Q^2}} = \frac{RI^2}{\sqrt{R^2 I^4 + \omega^2 L^2 I^4}} = \frac{R}{\sqrt{R^2 + \omega^2 L^2}}$$

3 Reactive power compensation (2)

- Reactive power compensation by adding capacitor in parallel to load
- Ideal compensation: reactive power Q_C produced by capacitor is equal to reactive power Q_L consumed by load



$$Q_C = -\omega C V^2, \quad Q_L = \Im\{\underline{S}_L\} = -BV^2 = \frac{\omega L}{R^2 + (\omega L)^2} V^2$$

$$\Rightarrow \quad C = \frac{L}{R^2 + (\omega L)^2}$$

3 Reactive power compensation (3)

- The approach on the previous slide only works perfectly if the load is constant, which is usually not the case

→ Compensation has to be adjusted with load variation!

- This can be done with so-called *capacitor banks*, that can insert or remove capacitors from the circuit by on-/off-switching of breakers
- For large loads with fast-varying demands (e.g. industrial loads), faster power-electronics based devices are needed
- Be careful not to overcompensate! That can also cause harm!

4 Outline

4 Conservation of complex apparent power

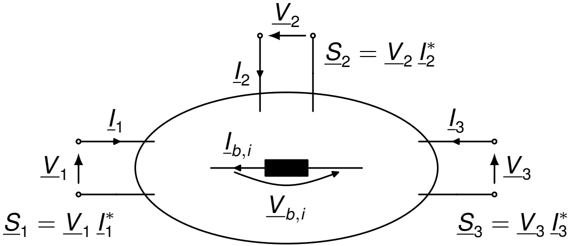
Theorem of conservation of complex apparent power

- Consider an electrical circuit with multiple sources and sinks that are all independent of each other
- Suppose that all voltages and currents in the circuit a purely sinusoidal and of the same frequency
- Then the sum of the apparent powers of the sources is equal to the sum of the apparent powers of the sinks

For single source, proof of the theorem follows from Kirchhoff's laws. For the general case, proof is more complicated.

4 Implications of theorem

- Helpful in analysis of large networks (for example: allows to replace complete networks by their Thevenin equivalents)



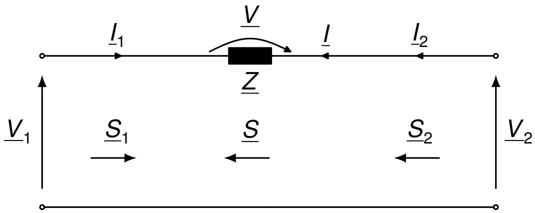
$$\underline{S}_1 + \underline{S}_2 + \underline{S}_3 = \sum_i \underline{S}_{b,i}$$

- Important implication for electric power systems:

  Sum of power injected in network
  = sum of consumption of all loads + sum of losses in all network elements

- Not obvious for reactive power!

4 Example: Conservation of complex apparent power



Task.
Given that $\underline{Z} = R + jX$, verify that $\underline{S}_1 + \underline{S}_2 = \underline{S}$!

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Solution.

Complex power at input-port 1

$$\underline{S}_1 = \underline{V}_1\,\underline{I}_1^* = P_1 + jQ_1$$

Complex currents

$$\underline{I}_1 = -\underline{I}_2 = -\underline{I}$$

Voltage at output-port 2

$$\underline{V}_2 = \underline{V}_1 + \underline{Z}\,\underline{I}$$

Complex power at output-port 2

$$\begin{aligned}\underline{S}_2 &= \underline{V}_2\,\underline{I}_2^* = \underline{V}_2\,\underline{I}^* = (\underline{V}_1 + \underline{Z}\,\underline{I})\underline{I}^* \\ &= \underline{V}_1\,\underline{I}^* + \underline{Z}|\underline{I}|^2 = \underbrace{(-P_1 + R|\underline{I}^2|)}_{P_2} + j\underbrace{(-Q_1 + X|\underline{I}|^2)}_{Q_2}\end{aligned}$$

Hence,

$$\underline{S}_1 + \underline{S}_2 = R|\underline{I}^2| + jX|\underline{I}|^2 = \underline{S}$$

$\rightarrow$ Input- and output-powers differ exactly by power consumed by impedance $\underline{Z}$

5 Outline

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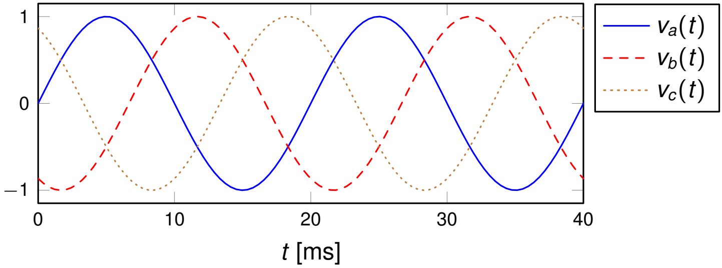
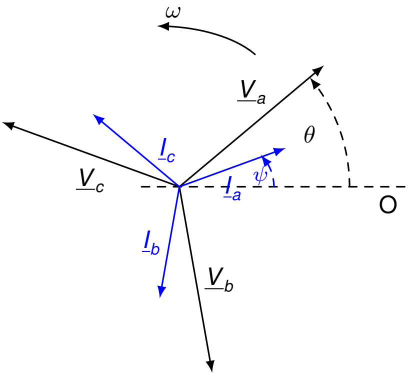
5.1 Balanced three-phase AC waveform

- Balanced three-phase AC waveform = 3-dimensional vector the elements of which are AC waveforms at the same frequency and with the same amplitude, but shifted by 120° (equivalently, $2\pi/3$ rad) with respect to each other
- Example: balanced three-phase voltage

$$v_{abc}(t) = \sqrt{2}V \begin{bmatrix} \sin(\omega t + \theta) \\ \sin\left(\omega t + \theta - \frac{2\pi}{3}\right) \\ \sin\left(\omega t + \theta - \frac{4\pi}{3}\right) \end{bmatrix}$$



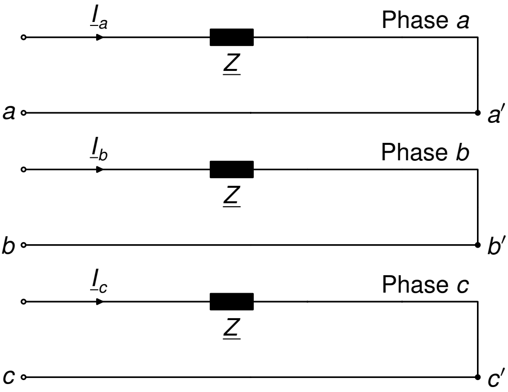
5.1 Phasor diagram balanced three-phase AC waveforms



- Observer in "O" sees voltage and current phasors rotating at speed ω passing in order *a*, *b*, *c*
- The phase sequence $a - b - c$ is called *positive* or *direct* sequence
- In balanced case: $\underline{V}_a + \underline{V}_b + \underline{V}_c = 0, \quad \underline{I}_a + \underline{I}_b + \underline{I}_c = 0$

5.1 Balanced three-phase circuit

- Balanced three-phase circuit = assembly of three identical circuits
- Each circuit is called a *phase*
- Example:



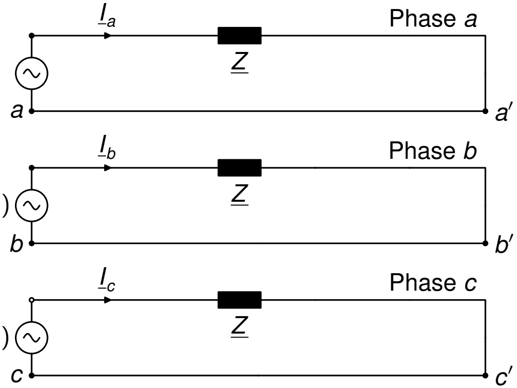
5.1 Balanced three-phase AC system

- Balanced three-phase AC system = balanced three-phase circuit, which is fed by balanced AC voltages (respectively currents)
- Three-phase AC systems are the predominant electrical systems used for power generation, transmission and distribution worldwide
- Example:

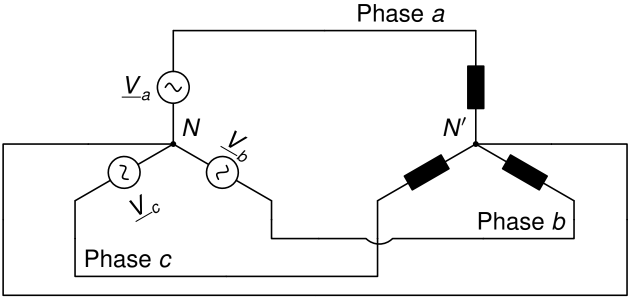
$$\underline{V}_a = V\angle(\theta)$$

$$\underline{V}_b = V\angle(\theta - \frac{2\pi}{3})$$

$$\underline{V}_c = V\angle(\theta - \frac{4\pi}{3})$$



- Common approach in practice: merge return conductors aa', bb', cc' into a single conductor
- The conductor $N - N'$ is called *neutral conductor* (Ουδέτερος αγωγός) and N and N' are called neutrals

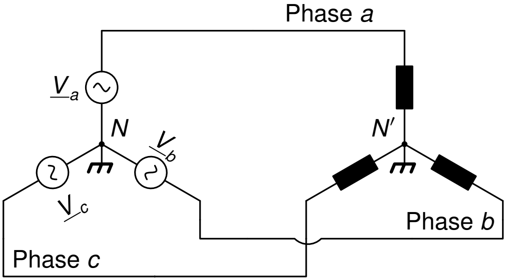


- Advantage: can transmit 3 times more current than in single-phase system with less than 2/3 of required conductor material of 3 single phase systems (4 instead of 6 and the neutral conductor has usually a smaller radius than the phase conductors)

5.2 Balanced Y-connection

- In balanced operation: all neutrals are at same voltage

→ Return conductor carries no current and is therefore often removed from circuit diagram ($\underline{I}_a + \underline{I}_b + \underline{I}_c = 0$)

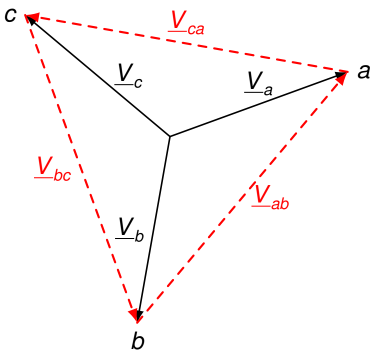


 Chassis ground = reference potential (not necessarily earth)

5.2 Phase and line voltages

- In three-phase systems, we can find *two different* types of voltages
  - Phase voltages between phase and neutral $\underline{V}_a$, $\underline{V}_b$, $\underline{V}_c$
  - Line voltages between different phases (lines) $\underline{V}_{ab}$, $\underline{V}_{bc}$, $\underline{V}_{ca}$
- The voltage indicated at the terminal of a three-phase element is the RMS value of the line voltage (unless otherwise specified)!

5.2 Phasor diagram for phase and line voltages



- Voltages between phases and neutral: $\underline{V}_a, \underline{V}_b, \underline{V}_c$
  → *phase-to-neutral* or *phase* voltages
- Voltages between individual phases:
  $\underline{V}_{ab} = \underline{V}_a - \underline{V}_b,\ \underline{V}_{bc} = \underline{V}_b - \underline{V}_c,\ \underline{V}_{ca} = \underline{V}_c - \underline{V}_a$
  → *line-to-line* or *line* voltages

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$$
\begin{aligned}
\underline{V}_{ab} &= \underline{V}_a - \underline{V}_b = V\angle(\theta) - V\angle(\theta - \frac{2\pi}{3}) = V\left(e^{j\theta} - e^{j(\theta - \frac{2\pi}{3})}\right)\\
&= V\left(\cos(\theta) + j\sin(\theta) - \left(\cos(\theta - \frac{2\pi}{3}) + j\sin(\theta - \frac{2\pi}{3})\right)\right)\\
&= V\Big(\cos(\theta) + j\sin(\theta) - \Big(\cos(\theta)\cos(\frac{-2\pi}{3}) - \sin(\theta)\sin(-\frac{2\pi}{3})\Big)\\
&\quad - j\Big(\sin(\theta)\cos(\frac{-2\pi}{3}) + \cos(\theta)\sin(\frac{-2\pi}{3})\Big)\Big)\\
&= V\left(\cos(\theta) + j\sin(\theta) - \left(-\frac{1}{2}\cos(\theta) + \frac{\sqrt{3}}{2}\sin(\theta)\right) - j\left(-\frac{1}{2}\sin(\theta) - \frac{\sqrt{3}}{2}\cos(\theta)\right)\right)\\
&= V\left(\frac{3}{2}\cos(\theta) - \frac{\sqrt{3}}{2}\sin(\theta) + j(\frac{3}{2}\sin(\theta) - \frac{\sqrt{3}}{2}\cos(\theta))\right)\\
&= \sqrt{3}V\left(\frac{\sqrt{3}}{2}\cos(\theta) - \frac{1}{2}\sin(\theta) + j(\frac{\sqrt{3}}{2}\sin(\theta) - \frac{1}{2}\cos(\theta))\right)\\
&= \sqrt{3}V\left(\cos(\frac{\pi}{6})\cos(\theta) - \sin(\frac{\pi}{6})\sin(\theta) + j(\cos(\frac{\pi}{6})\sin(\theta) - \sin(\frac{\pi}{6})\cos(\theta))\right)\\
&= \sqrt{3}V\left(\cos(\theta + \frac{\pi}{6}) + j\sin(\theta + \frac{\pi}{6})\right) = \sqrt{3}V\angle(\theta + \frac{\pi}{6})
\end{aligned}
$$

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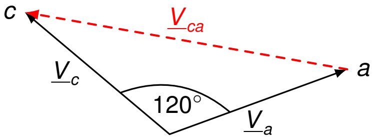
There is an easier way!

- From phasor diagram:



- Isosceles triangle (triangle that has two sides of equal length):

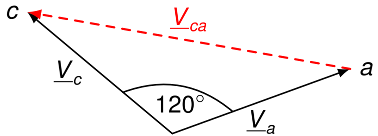
$$|\underline{V}_{ca}| = |\underline{V}_c - \underline{V}_a| = 2\sin\left(\frac{120}{2}\right)V = 2\frac{1}{2}\sqrt{3}V = \sqrt{3}V$$

⇒ In balanced case, for any line voltage $\underline{V}_{LL}$ and any phase voltage $\underline{V}_{LN}$, it holds that

$$|\underline{V}_{LL}| = \sqrt{3}|\underline{V}_{LN}|$$

(RMS value of line voltage = $\sqrt{3} \times$ RMS value of phase voltage)

- From phasor diagram:



- Sum of angles of a triangle=180° $\Rightarrow$ phase shift between $\underline{V}_c$ and $\underline{V}_{ca}$ is 30°
- Hence, we have the following relation between phase and line voltages

$$\underline{V}_{ab} = \sqrt{3}\underline{V}_a e^{j\frac{\pi}{6}}, \quad \underline{V}_{bc} = \sqrt{3}\underline{V}_b e^{j\frac{\pi}{6}}, \quad \underline{V}_{ca} = \sqrt{3}\underline{V}_c e^{j\frac{\pi}{6}}$$

- When specifying the voltage at the terminal of a three-phase device, unless otherwise specified, it is the effective (or RMS) value of the line voltages.

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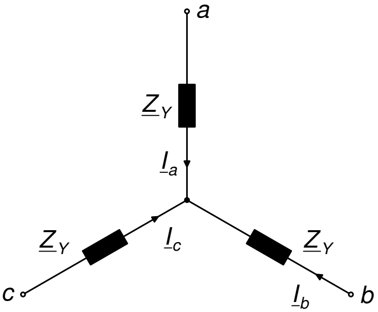
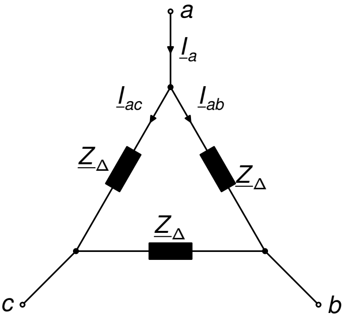
Y-connection



- Voltages across impedances are phase voltages $\underline{V}_a$, $\underline{V}_b$, $\underline{V}_c$
- Y-connection also called star-connection or wye-connection

Delta-connection



- Voltages across impedances are line voltages $\underline{V}_{ab}$, $\underline{V}_{bc}$, $\underline{V}_{ac}$
- Delta-connection also called Δ-connection

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- In Delta-connection

$$\underline{I}_a = \underline{I}_{ab} + \underline{I}_{ac} = \frac{\underline{V}_{ab} + \underline{V}_{ac}}{\underline{Z}_\Delta} = \frac{\underline{V}_{ab} - \underline{V}_{ca}}{\underline{Z}_\Delta}$$

- From phasor diagram: $\underline{V}_{ca}$ lags $\underline{V}_{ab}$ by 240°, or equivalently, $4\pi/3$ rad

$$\begin{aligned}\underline{I}_a &= \frac{\underline{V}_{ab} - \underline{V}_{ab} e^{-j\frac{4\pi}{3}}}{\underline{Z}_\Delta} = \frac{\underline{V}_{ab}}{\underline{Z}_\Delta}\left(1 - e^{-j\frac{4\pi}{3}}\right) \\ &= \frac{\underline{V}_{ab}}{\underline{Z}_\Delta}\left(1 - \cos\left(-\frac{4\pi}{3}\right) - \left(j\sin\left(-\frac{4\pi}{3}\right)\right)\right) \\ &= \frac{\underline{V}_{ab}}{\underline{Z}_\Delta}\left(1 + 0.5 - j0.5\sqrt{3}\right) = \frac{\underline{V}_{ab}}{\underline{Z}_\Delta}\sqrt{3}\left(\cos\left(-\frac{\pi}{6}\right) + j\sin\left(-\frac{\pi}{6}\right)\right) \\ &= \frac{\underline{V}_{ab}}{\underline{Z}_\Delta}\sqrt{3}e^{-j\frac{\pi}{6}} = \underline{I}_{ab}\sqrt{3}e^{-j\frac{\pi}{6}}\end{aligned}$$

$$\underline{I}_a = \underline{I}_{ab}\sqrt{3}e^{-j\frac{\pi}{6}}$$

⇒ In Delta-connection, the line currents $\underline{I}_a$, $\underline{I}_b$, $\underline{I}_c$ are $\sqrt{3}$ times higher than the load currents $\underline{I}_{ab}$, $\underline{I}_{ac}$, $\underline{I}_{bc}$ through the impedances $\underline{Z}_\Delta$ and lag by 30°, or equivalently, $\pi/6$ rad

5.2 Delta-Y-transformation

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- Loads may be connected either in Y- or Delta-connection
- For circuit analysis, usually one needs to transform Delta-connected loads to *equivalent* Y-connected loads
- Thereby, the RMS values of the phase currents $\underline{I}_a$, $\underline{I}_b$, $\underline{I}_c$ flowing into the load circuits have to remain the same if $\underline{V}_{ab}$, $\underline{V}_{bc}$, $\underline{V}_{ac}$ are the same
- In Delta-connection

$$\underline{I}_a = \frac{\underline{V}_{ab}\sqrt{3}e^{-j\frac{\pi}{6}}}{\underline{Z}_\Delta}$$

- In Y-connection

$$\underline{I}_a = \frac{\underline{V}_a}{\underline{Z}_Y} = \frac{\underline{V}_{ab}e^{-j\frac{\pi}{6}}}{\sqrt{3}\underline{Z}_Y}$$

$\Rightarrow \underline{Z}_\Delta = 3\underline{Z}_Y$

5.2 Practical remarks

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- Loads can be single- or three-phase loads (depending on their power demand)
- Single-phase loads can be connected either in branches of Y- or Delta-connection depending on their required voltage
  - In Cyprus: for most consumers, three-phase power supply at 400V (line voltage)/230V (phase voltage)
  - Appliances designed to work at 230V placed between a phase and neutral
- Houses may be connected in single- or three-phase (in Cyprus mostly single-phase)
- At one feeder, diverse single-phase appliances/houses connected to different phases so that overall the load is balanced as good as possible; this works never perfectly $\rightarrow$ nonzero neutral current
- However, as the number of loads increases, the phase currents increase and the neutral current becomes negligible

$\rightarrow$ From transmission network, most loads can be considered to be balanced

6 Outline

6 Instantaneous power in balanced three-phase systems (1)

- Balanced three-phase voltage and current

$$v_{abc}(t) = \sqrt{2}V \begin{bmatrix} \sin(\omega t) \\ \sin\left(\omega t - \frac{2\pi}{3}\right) \\ \sin\left(\omega t - \frac{4\pi}{3}\right) \end{bmatrix} \quad i_{abc}(t) = \sqrt{2}I \begin{bmatrix} \sin(\omega t - \varphi) \\ \sin\left(\omega t - \varphi - \frac{2\pi}{3}\right) \\ \sin\left(\omega t - \varphi - \frac{4\pi}{3}\right) \end{bmatrix}$$

- Instantaneous three-phase power

$$p(t) = v_a(t)i_a(t) + v_b(t)i_b(t) + v_c(t)i_c(t) = \sum_{i=a,b,c} p_i(t)$$

- From lecture on power in single-phase systems, we know that for $\underline{V}_a = V\underline{/0}$ and $\underline{I} = I\underline{/(}-\varphi)$ the instantaneous power in phase a is

$$\begin{aligned} p_a(t) &= VI\cos(\varphi)(1+\cos(2\omega t)) + VI\sin(\varphi)\sin(2\omega t) \\ &= P_a(1+\cos(2\omega t)) + Q_a\sin(2\omega t) \end{aligned}$$

6 Instantaneous power in balanced three-phase systems (2)

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$$p(t) = v_a(t)i_a(t) + v_b(t)i_b(t) + v_c(t)i_c(t)$$

$$\begin{aligned}
p(t) =& v_a(t)i_a(t) + v_b(t)i_b(t) + v_c(t)i_c(t) \\
=& 3VI\cos(\varphi) \\
&+ VI\cos(\varphi)\underbrace{\left(\cos(2\omega t) + \cos\left(2\left(\omega t - \frac{2\pi}{3}\right)\right) + \cos\left(2\left(\omega t - \frac{4\pi}{3}\right)\right)\right)}_{=0} \\
&+ VI\sin(\varphi)\underbrace{\left(\sin(2\omega t) + \sin\left(2\left(\omega t - \frac{2\pi}{3}\right)\right) + \sin\left(2\left(\omega t - \frac{4\pi}{3}\right)\right)\right)}_{=0} \\
&= 3VI\cos(\varphi) \\
&= 3P_a
\end{aligned}$$

- No fluctuating component in $p(t)$!
- This means that the oscillating components in each $p_i(t)$ compensate each other at each instant in time

6 Complex three-phase AC power

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- Under balanced conditions, the complex three-phase AC power is defined as

$$\begin{aligned}\underline{S}_{3\phi} &= \underline{V}_a\underline{I}_a^* + \underline{V}_b\underline{I}_b^* + \underline{V}_c\underline{I}_c^* \\ &= \underline{V}_a\underline{I}_a^* + \underline{V}_a e^{-j\frac{2\pi}{3}}\underline{I}_a^* e^{j\frac{2\pi}{3}} + \underline{V}_a e^{-j\frac{4\pi}{3}}\underline{I}_a^* e^{j\frac{4\pi}{3}} \\ &= 3\underline{V}_a\underline{I}_a^* \\ &= 3VIe^{j\varphi} \\ &= 3VI\cos(\varphi) + j3VI\sin(\varphi) \\ &= 3P_a + j3Q_a \text{ [VA]}\end{aligned}$$

- Three-phase active power: $P_{3\phi} = \Re\{\underline{S}_{3\phi}\} = 3VI\cos(\varphi) = 3P_a$ [W]
- Three-phase reactive power: $Q_{3\phi} = \Im\{\underline{S}_{3\phi}\} = 3VI\sin(\varphi) = 3Q_a$ [Var]

Under stationary and balanced conditions, total three-phase active power transmitted over a three-phase element is constant!

6 Complex three-phase AC power using line voltages

- Complex three-phase power

$$\underline{S}_{3\phi} = 3\underline{V}_{LN}\underline{I}_L^* = 3VIe^{j\varphi} = 3VI\cos(\varphi) + j3VI\sin(\varphi)$$

- With $\sqrt{3}\underline{V}_{LN} = \underline{V}_{LL}$ and $|\sqrt{3}\underline{V}_{LN}| = |\underline{V}_{LL}| = \sqrt{3}V = U$

$$\underline{S}_{3\phi} = \sqrt{3}\underline{V}_{LL}\underline{I}_L^* = \sqrt{3}UI\cos(\varphi) + j\sqrt{3}UI\sin(\varphi)$$

- These formulae are “hybrid” in so far as:
  - V_{LL} is the effective value of the *line voltage*
  - φ is the phase angle between the line current and the *phase-to-neutral voltage*.
- Three-phase active power: $P_{3\phi} = \Re\{\underline{S}_{3\phi}\} = \sqrt{3}UI\cos(\varphi)$
- Three-phase reactive power: $Q_{3\phi} = \Im\{\underline{S}_{3\phi}\} = \sqrt{3}UI\sin(\varphi)$

6 Remark on three-phase reactive power

- There is no oscillating component in $\underline{S}_{3\phi}$

$\rightarrow$ Three-phase reactive power $Q_{3\phi}$ is an *artificial* quantity

- Only single-phase reactive power has straightforward physical interpretation (in passive circuits)
- However, notion of three-phase reactive power used worldwide to establish analogy of three-phase complex power and single-phase complex power

6 Example: Power in balanced three-phase system

Task. Consider a three-phase load supplied by a 6 kV three-phase voltage source. Suppose the load current per phase is $\underline{I}_L = 2\angle(-10^\circ)$ A (phase shift compared to the voltage). Determine the power consumption of the load.

6 Example: Power in balanced three-phase system

Solution.

Complex three-phase power

$$\underline{S}_{3\phi} = 3\underline{V}_{LN}\underline{I}_L^* = 3VIe^{j\varphi} = 3VI\cos(\varphi) + j3VI\sin(\varphi)$$

In three-phase systems voltage amplitudes usually given for *line voltages*

Calculate phase voltage (choose voltage angle as 0°)

$$\underline{V}_{LN} = \frac{6}{\sqrt{3}}\underline{/0^\circ} = 3.46\text{ kV}$$

Thus,

$$\begin{aligned}\underline{S}_{3\phi} &= 3 \cdot 3.46 \cdot 2\underline{/10}^\circ = 20.76e^{j10^\circ} \\ &= 20.76\left(\cos(10^\circ) + j\sin(10^\circ)\right) = 20.44 + j3.60\text{ kVA}\end{aligned}$$

Note: alternatively, we could have used the power formula for the line voltages

$$\underline{S}_{3\phi} = \sqrt{3}\underline{V}_{LL}\underline{I}_L^*$$

7 Outline

7 Advantages of three-phase over 3 single-phase systems (1)

- Need less conductors:
  - 3 instead of 6 if no neutral conductor used (three-wire Y-connection)
  - 4 instead of 6 if neutral conductor is present (four-wire Y-connection, more common)
  - Neutral conductor used to reduce overvoltages (e.g., when switching lines on and off) and carry unbalanced currents (e.g., in case of single-phase short-circuit)
  - On transmission level: neutral currents small $\rightarrow$ can dimension neutral conductor much smaller than phase conductor
- Under balanced operation: No current flowing in return (neutral) conductor

  $\rightarrow$ Only half of line losses I^2R

  $\rightarrow$ Only half of line-voltage drop between source and load

7 Advantages of three-phase over 3 single-phase systems (2)

- Under balanced conditions, total instantaneous electrical power delivered by three-phase generator is (nearly) constant

$$p(t) = 3VI\cos(\varphi)$$

→ Also almost constant mechanical input

- Equation for instantaneous electrical power delivered by single-phase generator identical to that of instantaneous power in one phase:

$$p(t) = VI\cos(\varphi)(1 + \cos(2\omega t)) + VI\sin(\varphi)\sin(2\omega t)$$

- Double-frequency components create shaft vibration and noise

→ Could lead to failures in large machines

- Therefore most electric generators and loads rated > 5 kVA are constructed as three-phase machines

7 Summary

- Single-phase systems: power oscillates with 2ω, where ω is the stationary network frequency
- Balanced three-phase systems: oscillating power components in individual phases compensate each other $\rightarrow$ resulting three-phase power is constant over time
- Complex apparent power is product of voltage and complex conjugate current

$$\underline{S} = \underline{V}\,\underline{I}^* = VI\cos(\varphi) + jVI\sin(\varphi) = P + jQ$$

- Real part P of $\underline{S}$ is active power
- Imaginary part Q of $\underline{S}$ is reactive power
- For circuit calculations of single- and balanced three-phase systems, the same equations apply (per phase analysis)
- Three-phase complex apparent power under balanced conditions

$$\underline{S}_{3\phi} = 3\underline{V}\,\underline{I}^*$$